Approximate by Thinning: Deriving Fully Polynomial-Time Approximation Schemes Supplementary Proofs

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For the next few proofs, recall condition 3 of Theorem 6:

$$S^{\dagger} \circ \mathsf{F}\lceil Q\rceil^{\delta}_{\star} \subseteq \lceil Q\rceil^{\delta}_{+} \circ S^{\dagger}$$

1 Proofs Regarding the 0-1 Knapsack Problem

Let $S = [nil, S']$ where $S' = scons \cup snd$ and let $\preceq^{-}$ be defined by:

$$\begin{aligned} x \preceq^{\delta} y &\equiv x \leq^{\delta}_{v} y \wedge x \geq_{w} y \\ x \leq^{\delta}_{v} y &\equiv v\,x \leq \delta \cdot v\,y. \end{aligned}$$

We check that, letting $Q := (\preceq^{-})$, S and $\preceq^{-}$ satisfy condition 3. Note that $S^{\dagger} = [\langle nil, const\,1\rangle, \langle S' \circ (id \times fst), suc \circ snd \circ snd\rangle]$. Base case:

$$\begin{aligned} & \langle nil, const\,1\rangle \subseteq \lceil\preceq^{-}\rceil^{\delta}_{+} \circ \langle nil, const\,1\rangle \\ \equiv\ & ([\,],1) \leftsquigarrow \lceil\preceq^{-}\rceil^{\delta}_{+}\,([\,],1) \\ \equiv\ & \quad \{\ \text{definition of } \lceil\_\rceil^{\delta}_{+}\ \} \\ & ([\,],1) \preceq^{\delta^{0}} ([\,],1) \\ \equiv\ & \quad \{\ \text{definition of } \preceq^{\delta}\ \} \\ & [\,] \leq_{v} [\,] \wedge [\,] \geq_{w} [\,] \\ \equiv\ & true. \end{aligned}$$

For the non-empty case, let $S' = scons \cup snd$, we need that for $a :: Item$, $x_1, y_1, y_2 :: [Item]$, $n :: \mathbb{N}$, and $\delta :: \mathbb{R} \geq 1$:

$$\begin{aligned} & (x_1, n+1) \leftsquigarrow S'^{\dagger}\ (a,(y_1,n)) \wedge (y_1,n) \leftsquigarrow \lceil\preceq^{-}\rceil^{\delta}_{\star}\,(y_2,n) \Rightarrow \\ & \quad (\exists x_2 \mid: (x_1,n+1) \leftsquigarrow \lceil\preceq^{-}\rceil^{\delta}_{+}\,(x_2,n+1) \wedge \\ & \qquad\qquad (x_2,n+1) \leftsquigarrow S'^{\dagger}\ (a,(y_2,n))). \end{aligned}$$

Expanding the liftings, we get

$$x_1 \leftsquigarrow S' \ (a, y_1) \ \wedge \ y_1 \preceq^{\delta^n} y_2 \ \Rightarrow$$
$$(\exists x_2 \mid : x_1 \preceq^{\delta^n} x_2 \ \wedge \ x_2 \leftsquigarrow S' \ (a, y_2)).$$

Case: $x_1 = y_1$. Choose $x_2 = y_2$. Using the assumption we immediately have $x_1 = y_1 \preceq^{\delta^n} y_2 = x_2$.

Case: $x_1 = a : y_1$. Choose $x_2 = a : y_2$. We then have to show that $a : y_1 \preceq^{\delta^n} a : y_2$, that is, $a : y_1 \leq_v^{\delta^n} a : y_2 \wedge a : y_1 \geq_w a : y_2$. To show that $a : y_1 \leq_v^{\delta^n} a : y_2$, we reason:

$$\begin{aligned}
& \mathit{value}\ (a : y_1) \\
= \ & \mathit{value}\ a + \mathit{value}\ y_1 \\
\leq \ & \quad \{ \ \text{since } y_1 \preceq^{\delta^n} y_2 \Rightarrow y_1 \leq^{\delta^n} y_2 \ \} \\
& \mathit{value}\ a + \delta^n \cdot \mathit{value}\ y_2 \\
\leq \ & \quad \{ \ \text{since } \delta \geq 1 \ \} \\
& \delta^n \cdot \mathit{value}\ a + \delta^n \cdot \mathit{value}\ y_2 \\
= \ & \delta^n \cdot \mathit{value}\ (a : y_2).
\end{aligned}$$

To show that $a : y_1 \geq_w a : y_2$, we reason:

$$\begin{aligned}
& \mathit{weight}\,(a : y_1) \\
= \ & \mathit{weight}\, a + \mathit{weight}\, y_1 \\
\geq \ & \quad \{ \ \text{since } y_1 \preceq^{\delta^n} y_2 \Rightarrow y_1 \geq_w y_2 \ \} \\
= \ & \mathit{weight}\, a + \mathit{weight}\, y_2 \\
= \ & \mathit{weight}\,(a : y_2).
\end{aligned}$$

Note that, since $\mathit{weight}\ (a : y_1) \leq W$, the above also implies that $\mathit{weight}\ (a : y_2) \leq W$.

2 Proofs Regarding Minimising Total Late Work

Let $S = [\langle \mathit{nil}, \mathit{nil} \rangle, S']$ where $S' = \mathit{work} \cup \mathit{drop}$ and let $\succeq^{\text{-}}$ be defined by:

$$\begin{aligned}
(x_1, y_1) \succeq^{\delta} (x_2, y_2) \quad \equiv \quad & (x_1, y_1) \geq_p^{\delta} (x_2, y_2) \ \wedge \\
& \mathit{totalspan}\ x_1 \geq \mathit{totalspan}\ x_2,
\end{aligned}$$

We check that S and $Q := \succeq^{\text{-}}$ satisfies condition 3. Note that

$$S^{\dagger} = [\langle \langle \mathit{nil}, \mathit{nil} \rangle, \mathit{const}\, 2 \rangle, \langle S' \circ (\mathit{fst} \times \mathit{id}), \mathit{suc} \circ \mathit{snd} \circ \mathit{fst} \rangle].$$

Base case:

$$\langle \langle \mathit{nil}, \mathit{nil} \rangle, \mathit{const}\, 2 \rangle \ \subseteq \ \lceil \preceq^{\text{-}} \rceil_{+}^{\delta} \circ \langle \langle \mathit{nil}, \mathit{nil} \rangle, \mathit{const}\, 2 \rangle$$

$$
\begin{aligned}
&\equiv \;(([\,],[\,]),2) \leftsquigarrow \lceil \succeq^{-} \rceil^{\delta}_{+} \,(([\,],[\,]),2)\\
&\equiv \quad \{\ \text{definition of } \lceil \text{-} \rceil^{\delta}_{+}\ \}\\
&\quad ([\,],1) \succeq^{\delta^2} ([\,],1)\\
&\equiv \quad \{\ \text{definition of } \succeq^{-}\ \}\\
&\quad ([\,],[\,]) \geq_p^{\delta^2} ([\,],[\,]) \;\wedge\; \mathit{totalspan}\,[\,] \geq \mathit{totalspan}\,[\,]\\
&\equiv \;\mathit{true}.
\end{aligned}
$$

The inductive case simplifies to that for all $zw_1, xy_1, xy_2 :: \mathit{Sche}$:

$$
\begin{aligned}
&zw_1 \leftsquigarrow S'\ (xy_1, a) \;\wedge\; xy_1 \succeq^{\delta^n} xy_2 \;\Rightarrow\\
&\quad (\exists zw_2 \mid: zw_1 \succeq^{\delta^n} zw_2 \;\wedge\; zw_2 \leftsquigarrow S'\ (xy_2, a)).
\end{aligned}
$$

Let $xy_1 = (x_1, y_1) \succeq^{\delta} (x_2, y_2) = xy_2$.

Case $zw_1 = \mathit{work}\ ((x_1, y_1), a) = (x_1 \triangleright a, y_1)$. We choose $zw_2 = (x_2 \triangleright a, y_2)$. Certainly, $\mathit{totalspan}\ (x_1 \triangleright a) \geq \mathit{totalspan}\ (x_2 \triangleright a)$. To show that $zw_1 \geq_p^{\delta^n} zw_2$ we reason (abbreviating *penalty* to p):

$$
\begin{aligned}
&\quad \delta^n \cdot p\ (x_1 \triangleright a, y_1)\\
&= \delta^n \cdot ((\mathit{span}\ a + \mathit{totalspan}\ x_1 - \mathit{due}\ a) \uparrow 0) + \delta^n \cdot p\ (x_1, y_1)\\
&\geq \quad \{\ \text{since } \delta^n \cdot p\ (x_1, y_1) \geq p(x_2, y_2)\ \}\\
&\quad \delta^n \cdot ((\mathit{span}\ a + \mathit{totalspan}\ x_1 - \mathit{due}\ a) \uparrow 0) + p\ (x_2, y_2)\\
&\geq \quad \{\ \text{since } \mathit{totalspan}\ x_1 \geq \mathit{totalspan}\ x_2\ \}\\
&\quad \delta^n \cdot ((\mathit{span}\ a + \mathit{totalspan}\ x_2 - \mathit{due}\ a) \uparrow 0) + p\ (x_2, y_2)\\
&> \quad \{\ \text{since } \delta > 1\ \}\\
&\quad ((\mathit{span}\ a + \mathit{totalspan}\ x_2 - \mathit{due}\ a) \uparrow 0) + p\ (x_2, y_2)\\
&= p\ (x_2 \triangleright a, y_2).
\end{aligned}
$$

Case $zw_1 = \mathit{drop}\ ((x_1, y_1), a) = (x_1, y_1 \triangleright a)$. We choose $zw_2 = (x_2, y_2 \triangleright a)$ It is still true that $\mathit{totalspan}\ x_1 \geq \mathit{totalspan}\ x_2$. To show that $zw_1 \geq_p^{\delta^n} zw_2$ we reason:

$$
\begin{aligned}
&\quad \delta^n \cdot p\ (x_1, y_1 \triangleright a)\\
&= \delta^n \cdot (\mathit{delays}\, x_1 + \mathit{totalspan}\, y_1 + \mathit{span}\, a)\\
&\geq \quad \{\ \text{since } xy_1 \geq_p^{\delta^n} xy_2\ \}\\
&= \mathit{delays}\, x_2 + \mathit{totalspan}\, y_2 + \delta^n \cdot \mathit{span}\, a\\
&\geq \quad \{\ \text{since } \delta > 1\ \}\\
&= \mathit{delays}\, x_2 + \mathit{totalspan}\, y_2 + \mathit{span}\, a\\
&= p\ (x_2, y_2 \triangleright a).
\end{aligned}
$$

3 Proofs Regarding the Tree Partitioning Problem

Let $(\preceq^{\delta}) \ = \ (\preceq^{\delta}_{\mathsf{s}}) \cup (\preceq^{\delta}_{\mathsf{d}})$, where $\preceq^{\delta}_{\mathsf{s}}$ and $\preceq^{\delta}_{\mathsf{d}}$ are defined by:

$$\begin{aligned}(t, ts) \preceq^{\delta}_{\mathsf{s}} (u, us) \ &\equiv \ \mathit{existsS}\ t \wedge \mathit{existsS}\ u \ \wedge \\ &\qquad (t, ts) \leq^{\delta}_{\mathit{ff}} (u, us) \ \wedge \ t \leq_m u, \\ (t, ts) \preceq^{\delta}_{\mathsf{d}} (u, us) \ &\equiv \ \mathit{allD}\ t \wedge \mathit{allD}\ u \ \wedge \\ &\qquad (t, ts) \preceq^{\delta}_{ex} (u, us) \ \wedge \ t \geq_d u.\end{aligned}$$

Furthermore, $\preceq^{\delta}_{ex}$ is defined by:

$$\begin{aligned}&(t, ts) \preceq^{\delta}_{ex} (u, us) \ \equiv \\ &\qquad \mathit{demand}\, t + \mathit{fulfilment}\ ts \leq \mathit{demand}\, u + \delta \cdot \mathit{fulfilment}\ ts.\end{aligned}$$

Lemma 1.

$$(t, ts) \preceq^{\delta}_{\mathsf{d}} (u, us) \ \Rightarrow \ ts \leq^{\delta}_{\mathit{ff}} us \ \Rightarrow \ (t, ts) \leq^{\delta}_{\mathit{ff}} (u, us).$$

Proof. We abbreviate *demand* to d. To prove the first implication:

$$\begin{aligned}&\quad (t, ts) \preceq^{\delta}_{\mathsf{d}} (u, us) \\ \Rightarrow &\quad (t, ts) \preceq^{\delta}_{ex} (u, us) \ \wedge \ t \geq_d u \\ \equiv &\quad d\,t + \mathit{ff}\ ts \leq d\,u + \delta \cdot \mathit{ff}\ us \ \wedge \ t \geq_d u \\ \Rightarrow &\qquad \{ \ \text{using } t \geq_d u \ \} \\ &\quad d\,t + \mathit{ff}\ ts \leq d\,t + \delta \cdot \mathit{ff}\ us \\ \equiv &\quad \mathit{ff}\ ts \leq \delta \cdot \mathit{ff}\ us.\end{aligned}$$

Note that the penultimate step would not go through had the righthand side be $\delta \cdot (d\,u + \mathit{ff}\ us)$. The second implication is immediate since $\mathit{ff}\ t = \mathit{ff}\ u = 0$. □

The condition 3 in Theorem 6 we need to show is that for all $tts_1, uus_i, vvs_i :: \mathit{Part}$ (where $i \in \{1, 2\}$), and $m, n : \mathbb{N}$, we have:

$$\begin{aligned}&tts_1 \leftsquigarrow S\ (uus_1, vvs_1) \ \wedge \ uus_1 \preceq^{\delta^m} uus_2 \ \wedge \ vvs_1 \preceq^{\delta^n} vvs_2 \\ &\quad \Rightarrow (\exists tts_2 : tts_1 \preceq^{\delta^{m+n}} tts_2 \ \wedge \ tts_2 \leftsquigarrow S\ (uus_2, vvs_2)),\end{aligned}$$

where $S = \mathit{felink} \cup \mathit{sep}$. Note the superscript: if uus_2 subsumes uus_1 by δ^m and vvs_2 subsumes vvs_1 by δ^n, we want tts_2 to subsume tts_1 by δ^{m+n}.

Let $uus_i = (u_i, us_i)$ and similarly for tts_i and vvs_i, for $i \in \{1, 2\}$.

Case 1. $\mathit{allD}\ u_i$ and $\mathit{allD}\ v_i$. That is, $uus_1 \preceq^{\delta^m}_{\mathsf{d}} uus_2 \ \wedge \ vvs_1 \preceq^{\delta^n}_{\mathsf{d}} vvs_2$. We further distinguish between two cases:

Case 1.1. $tts_1 = \mathit{sep}\,(uus_1, vvs_1) = (v_1, \wr u_1 \int \cup us_1 \cup vs_1)$.

For this case we pick $tts_2 = (v_2, \wr u_2 \int \cup us_2 \cup vs_2)$. The task is to show that $tts_1 \preceq_{\mathsf{d}}^{\delta^{m+n}} tts_2$, which expands to

$$tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2 \ \wedge\ v_1 \geq_d v_2.$$

To show that $tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2$, we reason:

$$\begin{array}{ll} & d\, v_1 + \mathit{ff}\ u_1 + \mathit{ff}\ us_1 + \mathit{ff}\ vs_1 \\ \leq & \{\ \text{since } uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2 \text{ and thus } uus_1 \leq_{\mathit{ff}}^{\delta^m} uus_2\ \} \\ & d\, v_1 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ us_2 + \mathit{ff}\ vs_1 \\ \leq & \{\ \text{since } vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2 \text{ and thus } vvs_1 \leq_{ex}^{\delta^n} vvs_2\ \} \\ & d\, v_2 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ us_2 + \delta^n \cdot \mathit{ff}\ vs_2 \\ \leq & \{\ \text{since } \delta \geq 1\ \} \\ & d\, v_2 + \delta^{m+n} \cdot (\mathit{ff}\ u_2 + \mathit{ff}\ us_2 + \mathit{ff}\ vs_2). \end{array}$$

That $v_1 \geq_d v_2$ follows from $vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2$.

Case 1.2. $tts_1 = \mathit{felink}\,(uus_1, vvs_1) = (\mathsf{N}\,a\ (u_1{:}ws_1), us_1{\cup}vs_1)$, where $\mathsf{N}\,a\,ws_1 = v_1$. For this case, we pick $tts_2 = (\mathsf{N}\,a\ (u_2 : ws_2), us_2 \cup vs_2)$, where $\mathsf{N}\,a\,ws_2 = v_2$. Since none of u_i and v_i contain supply nodes, both of the new roots are feasible.

The task is to show that $tts_1 \preceq_{\mathsf{d}}^{\delta^{m+n}} tts_2$, which expands to

$$tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2 \ \wedge\ \mathsf{N}\,a\ (u_1 : ws_1) \geq_d \mathsf{N}\,a\ (u_2 : ws_2).$$

For $tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2$ we reason:

$$\begin{array}{ll} & d\,(\mathsf{N}\,a\ (u_1 : ws_1)) + \mathit{ff}\ us_1 + \mathit{ff}\ vs_1 \\ = & d\,a + d\,u_1 + d\,ws_1 + \mathit{ff}\ us_1 + \mathit{ff}\ vs_1 \\ = & d\,v_1 + \mathit{ff}\ vs_1 + d\,u_1 + \mathit{ff}\ us_1 \\ \leq & \{\ \text{since } uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2 \text{ and thus } uus_1 \preceq_{ex}^{\delta^m} uus_2\ \} \\ & d\,v_1 + \mathit{ff}\ vs_1 + d\,u_2 + \delta^m \cdot \mathit{ff}\ us_2 \\ \leq & \{\ \text{since } vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2 \text{ and thus } vvs_1 \preceq_{ex}^{\delta^n} vvs_2\ \} \\ & d\,v_2 + \delta^n \cdot \mathit{ff}\ vs_2 + d\,u_2 + \delta^m \cdot \mathit{ff}\ us_2 \\ \leq & \{\ \text{since } \delta \geq 1\ \} \\ & d\,v_2 + \delta^{m+n} \cdot \mathit{ff}\ vs_1 + d\,u_2 + \delta^{m+n} \cdot \mathit{ff}\ us_2 \\ = & d\,(\mathsf{N}\,a\ (u_2 : ws_2)) + \delta^{m+n} \cdot \mathit{ff}\ us_2 + \mathit{ff}\ vs_2. \end{array}$$

For $\mathsf{N}\,a\ (u_1 : ws_1) \geq_d \mathsf{N}\,a\ (u_2 : ws_2)$ we reason:

$$\begin{array}{ll} & d\,(\mathsf{N}\,a\ (u_1 : ws_1)) \\ = & d\,a + d\,u_1 + d\,ws_1 \\ = & d\,v_1 + d\,u_1 \end{array}$$

$$\begin{aligned}
&\geq \quad \{ \text{ since } uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2 \text{ and } vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2 \ \} \\
&= \ d\,v_2 + d\,u_2 \\
&= \ d\,(\mathsf{N}\,a\ (u_2 : ws_2)).
\end{aligned}$$

Case 2: *existsS* u_i and *allD* v_i. That is, $uus_1 \preceq_{\mathsf{s}}^{\delta^m} uus_2$ and $vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2$. We further distinguish between two possible sub-cases:

Case 2.1. tts_1 is generated by *sep*, that is, $tts_1 = (v_1, \wr u_1 \int \cup us_1 \cup vs_1)$. For this case we also pick $tts_2 = (v_2, \wr u_2 \int \cup us_2 \cup vs_2)$. The task is to show that $tts_1 \preceq_{\mathsf{d}}^{\delta^{m+n}} tts_2$, which expands to

$$tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2 \ \wedge \ v_1 \geq_d v_2.$$

For $tts_1 \preceq_{ex}^{\delta^{m+n}} tts_2$ we reason:

$$\begin{aligned}
& d\ v_1 + \mathit{ff}\ u_1 + \mathit{ff}\ us_1 + \mathit{ff}\ vs_1 \\
\leq &\quad \{ \text{ since } uus_1 \preceq_{\mathsf{s}}^{\delta^m} uus_2 \text{ and thus } uus_1 \leq_{\mathit{ff}}^{\delta^m} uus_2 \ \} \\
& d\ v_1 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ us_2 + \mathit{ff}\ vs_1 \\
\leq &\quad \{ \text{ since } vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2 \text{ and thus } vvs_1 \leq_{ex}^{\delta^n} vvs_2 \ \} \\
& d\ v_2 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ us_2 + \delta^n \cdot \mathit{ff}\ vs_2 \\
\leq &\quad \{ \text{ since } \delta \geq 1 \ \} \\
& d\ v_2 + \delta^{m+n} \cdot (\mathit{ff}\ u_2 + \mathit{ff}\ us_2 + \mathit{ff}\ vs_2).
\end{aligned}$$

That $v_1 \geq_d v_2$ follows immediately from $vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2$.

Case 2.2. tts_1 is generated by *felink*, that is, v_1 can be decomposed into $\mathsf{N}\,a\,ws_1$, and $tts_1 = (\mathsf{N}\,a\ (u_1 : ws_1), us_1 \cup vs_1)$. For this partition to be feasible, we must have *margin* $u_1 \geq$ *demand* v_1.

For this case we also decompose v_2 into $\mathsf{N}\,a\,ws_2$, and pick $tts_2 = (\mathsf{N}\,a\ (u_2 : ws_2), us_2 \cup vs_2)$.

We have to show that $\mathsf{N}\,a\ (u_2 : ws_2)$ is feasible, that is

$$\mathit{margin}\ u_2 \geq \mathit{demand}\ v_2. \tag{1}$$

Then we show that $tts_1 \preceq_{\mathsf{s}}^{\delta^{m+n}} tts_2$, which expands to

$$\begin{aligned}
& tts_1 \leq_{\mathit{ff}}^{\delta^{m+n}} tts_2 \ \wedge & (2)\\
& \mathsf{N}\,a\ (u_1 : ws_1) \leq_m \mathsf{N}\,a\ (u_2 : ws_2). & (3)
\end{aligned}$$

For (1), we reason:

$$\begin{aligned}
& \mathit{margin}\,u_2 \\
\geq &\quad \{ \text{ since } uus_1 \preceq_{\mathsf{s}}^{\delta^m} uus_2 \ \}
\end{aligned}$$

$$
\begin{aligned}
& \mathit{margin}\, u_1 \\
\geq\; & \quad \{ \text{ since } \mathsf{N}\, a\ (u_1 : ws_1) \text{ is feasible } \} \\
& \mathit{demand}\, v_1 \\
\geq\; & \quad \{ \text{ since } vvs_1 \preceq_{\mathsf{d}}^{\delta^n} vvs_2 \ \} \\
& \mathit{demand}\, v_2 .
\end{aligned}
$$

For (3), we reason:

$$
\begin{aligned}
& \mathit{margin}\, (\mathsf{N}\, a\ (u_1 : ws_1)) \\
=\; & \quad \{ \text{ the supply node is in } u_1 \ \} \\
& \mathit{margin}\, u_1 - \mathit{demand}\, v_1 \\
\leq\; & \quad \{ \text{ since } u_1 \leq_m u_2 \text{ and } v_1 \geq_m v_2 \ \} \\
& \mathit{margin}\, u_2 - \mathit{demand}\, v_2 \\
=\; & \quad \{ \text{ the supply node is in } u_2 \ \} \\
& \mathit{margin}\, (\mathsf{N}\, a\ (u_2 : ws_2)).
\end{aligned}
$$

To show (2), we reason:

$$
\begin{aligned}
& \mathit{fulfilment}\ tts_1 \\
=\; & \mathit{ff}\ u_1 + \mathit{demand}\ v_1 + \mathit{ff}\ us_1 + \mathit{ff}\ vs_1 \\
\leq\; & \quad \{ \text{ since } uus_1 \leq_{\mathit{ff}}^{\delta^m} uus_2 \ \} \\
& \delta^m \cdot \mathit{ff}\ u_2 + \mathit{demand}\ v_1 + \delta^m \cdot \mathit{ff}\ us_2 + \mathit{ff}\ vs_1 \\
\leq\; & \quad \{ \text{ since } vvs_1 \preceq_{ex}^{\delta^n} vvs_2 \ \} \\
& \delta^m \cdot \mathit{ff}\ u_2 + \mathit{demand}\ v_2 + \delta^m \cdot \mathit{ff}\ us_2 + \delta^n \cdot \mathit{ff}\ vs_2 \\
\leq\; & \quad \{ \text{ since } \delta \geq 1 \ \} \\
& \delta^{m+n} \cdot (\mathit{ff}\ u_2 + \mathit{demand}\ v_2 + \mathit{ff}\ us_2 + \mathit{ff}\ vs_2) \\
=\; & \delta^{m+n} \cdot \mathit{fulfilment}\ tts_2 .
\end{aligned}
$$

Case 3: $\mathit{allD}\ u_i$ and $\mathit{existsS}\ v_i$. That is, $uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2 \ \wedge\ vvs_1 \preceq_{\mathsf{s}}^{\delta^n} vvs_2$. We further distinguish between two possible sub-cases:

Case 3.1. tts_1 is generated by *sep*, that is, $tts_1 = (v_1, \wr u_1 \int \cup\, us_1 \cup vs_1)$. For this case we also pick $tts_2 = (v_2, \wr u_2 \int \cup\, us_2 \cup vs_2)$. The task is to show that $tts_1 \preceq_{\mathsf{s}}^{\delta^{m+n}} tts_2$, which expands to

$$tts_1 \leq_{\mathit{ff}}^{\delta^{m+n}} tts_2 \ \wedge\ v_1 \leq_m v_2 .$$

For $tts_1 \leq_{\mathit{ff}}^{\delta^{m+n}} tts_2$ we reason:

$$
\begin{aligned}
& \mathit{ff}\, v_1 + \mathit{ff}\, u_1 + \mathit{ff}\, us_1 + \mathit{ff}\, vs_1 \\
\leq\; & \quad \{ \text{ since } vvs_1 \preceq_{\mathsf{s}}^{\delta^n} vvs_2 \text{ and thus } vvs_1 \leq_{\mathit{ff}}^{\delta^n} vvs_2 \ \}
\end{aligned}
$$

$$
\begin{array}{ll}
& \delta^n \cdot \mathit{ff}\ v_2 + \mathit{ff}\ u_1 + \mathit{ff}\ us_1 + \delta^n \cdot \mathit{ff}\ vs_2 \\
\leq & \{\ \text{since } uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2 \text{ and thus } uus_1 \leq_{\mathit{ff}}^{\delta^m} uus_2\ \} \\
& \delta^n \cdot \mathit{ff}\ v_2 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ us_2 + \delta^n \cdot \mathit{ff}\ vs_2 \\
\leq & \{\ \text{since } \delta \geq 1\ \} \\
& \delta^{m+n} \cdot (\mathit{ff}\ v_2 + \mathit{ff}\ u_2 + \mathit{ff}\ us_2 + \mathit{ff}\ vs_2).
\end{array}
$$

That $v_1 \leq_m v_2$ follows immediately from $vvs_1 \preceq_{\mathsf{s}}^{\delta^n} vvs_2$.

Case 3.2. tts_1 is generated by *felink*, that is, v_1 can be decomposed into $\mathsf{N}\,a\,ws_1$, and $tts_1 = (\mathsf{N}\,a\ (u_1 : ws_1), us_1 \cup vs_1)$. For this partition to be feasible, we must have $\mathit{margin}\ v_1 \geq \mathit{demand}\ u_1$.

For this case we also decompose v_2 into $\mathsf{N}\,a\,ws_2$, and pick $tts_2 = (\mathsf{N}\,a\ (u_2 : ws_2), us_2 \cup vs_2)$.

We have to show that $\mathsf{N}\,a\ (u_2 : ws_2)$ is feasible, that is

$$\mathit{margin}\ v_2 \geq \mathit{demand}\ u_2. \tag{4}$$

Then we show that $tts_1 \preceq_{\mathsf{s}}^{\delta^{m+n}} tts_2$, which expands to

$$
\begin{array}{ll}
tts_1 \leq_{\mathit{ff}}^{\delta^{m+n}} tts_2\ \ \wedge & (5) \\
\mathsf{N}\,a\ (u_1 : ws_1) \leq_m \mathsf{N}\,a\ (u_2 : ws_2). & (6)
\end{array}
$$

For (4), we reason:

$$
\begin{array}{ll}
& \mathit{margin}\, v_2 \\
\geq & \{\ \text{since } vvs_1 \preceq_{\mathsf{s}}^{\delta^n} vvs_2\ \} \\
& \mathit{margin}\, v_1 \\
\geq & \{\ \text{since } \mathsf{N}\,a\ (u_1 : ws_1) \text{ is feasible}\ \} \\
& \mathit{demand}\, u_1 \\
\geq & \{\ \text{since } uus_1 \preceq_{\mathsf{d}}^{\delta^m} uus_2\ \} \\
& \mathit{demand}\, u_2.
\end{array}
$$

For (6), we reason:

$$
\begin{array}{ll}
& \mathit{margin}\, (\mathsf{N}\,a\ (u_1 : ws_1)) \\
= & \{\ \text{the supply node is in } v_1\ \} \\
& \mathit{margin}\, v_1 - \mathit{demand}\, u_1 \\
\leq & \{\ \text{since } u_1 \leq_m u_2 \text{ and } v_1 \geq_m v_2\ \} \\
& \mathit{margin}\, v_2 - \mathit{demand}\, 2_2 \\
= & \{\ \text{the supply node is in } v_2\ \} \\
& \mathit{margin}\, (\mathsf{N}\,a\ (u_2 : ws_2)).
\end{array}
$$

To show (5), we reason:

$$
\begin{array}{ll}
& \mathit{fulfilment}\ \mathit{tts}_1 \\
= & \mathit{ff}\ v_1 + \mathit{demand}\ u_1 + \mathit{ff}\ \mathit{us}_1 + \mathit{ff}\ \mathit{vs}_1 \\
\leq & \quad \{\ \text{since } \mathit{vvs}_1 \leq_{\mathit{ff}}^{\delta^n} \mathit{vvs}_2\ \} \\
& \delta^n \cdot \mathit{ff}\ v_2 + \mathit{demand}\ u_1 + \mathit{ff}\ \mathit{us}_1 + \delta^n \cdot \mathit{ff}\ \mathit{vs}_1 \\
\leq & \quad \{\ \text{since } \mathit{uus}_1 \preceq_{ex}^{\delta^n} \mathit{uus}_2\ \} \\
& \delta^n \cdot \mathit{ff}\ v_2 + \mathit{demand}\ u_2 + \delta^m \cdot \mathit{ff}\ \mathit{us}_2 + \delta^n \cdot \mathit{ff}\ \mathit{vs}_2 \\
\leq & \quad \{\ \text{since } \delta \geq 1\ \} \\
& \delta^{m+n} \cdot (\mathit{ff}\ v_2 + \mathit{demand}\ u_2 + \mathit{ff}\ \mathit{us}_2 + \mathit{ff}\ \mathit{vs}_2) \\
= & \delta^{m+n} \cdot \mathit{fulfilment}\ \mathit{tts}_2.
\end{array}
$$

Case 4: *existsS* u_i and *existsS* v_i. That is, $\mathit{uus}_1 \preceq_{\mathsf{s}}^{\delta^m} \mathit{uus}_2 \ \wedge\ \mathit{vvs}_1 \preceq_{\mathsf{s}}^{\delta^n} \mathit{vvs}_2$. For this case the only possibility is to generate tts_1 by *sep*, that is, $\mathit{tts}_1 = (v_1, \wr u_1 \int \cup\, \mathit{us}_1 \cup \mathit{vs}_1)$. We also pick $\mathit{tts}_2 = (v_2, \wr u_2 \int \cup\, \mathit{us}_2 \cup \mathit{vs}_2)$. The task is to show that $\mathit{tts}_1 \preceq_{\mathsf{s}}^{\delta^{m+n}} \mathit{tts}_2$, which expands to

$$
\mathit{tts}_1 \leq_{\mathit{ff}}^{\delta^{m+n}} \mathit{tts}_2 \ \wedge\ v_1 \leq_m v_2.
$$

For $\mathit{tts}_1 \leq_{\mathit{ff}}^{\delta^{m+n}} \mathit{tts}_2$ we reason:

$$
\begin{array}{ll}
& \mathit{ff}\ v_1 + \mathit{ff}\ u_1 + \mathit{ff}\ \mathit{us}_1 + \mathit{ff}\ \mathit{vs}_1 \\
\leq & \quad \{\ \text{since } \mathit{vvs}_1 \preceq_{\mathsf{s}}^{\delta^n} \mathit{vvs}_2 \text{ and thus } \mathit{vvs}_1 \leq_{\mathit{ff}}^{\delta^n} \mathit{vvs}_2\ \} \\
& \delta^n \cdot \mathit{ff}\ v_2 + \mathit{ff}\ u_1 + \mathit{ff}\ \mathit{us}_1 + \delta^n \cdot \mathit{ff}\ \mathit{vs}_2 \\
\leq & \quad \{\ \text{since } \mathit{uus}_1 \preceq_{\mathsf{s}}^{\delta^m} \mathit{uus}_2 \text{ and thus } \mathit{uus}_1 \leq_{\mathit{ff}}^{\delta^m} \mathit{uus}_2\ \} \\
& \delta^n \cdot \mathit{ff}\ v_2 + \delta^m \cdot \mathit{ff}\ u_2 + \delta^m \cdot \mathit{ff}\ \mathit{us}_2 + \delta^n \cdot \mathit{ff}\ \mathit{vs}_2 \\
\leq & \quad \{\ \text{since } \delta \geq 1\ \} \\
& \delta^{m+n} \cdot (\mathit{ff}\ v_2 + \mathit{ff}\ u_2 + \mathit{ff}\ \mathit{us}_2 + \mathit{ff}\ \mathit{vs}_2).
\end{array}
$$

That $v_1 \leq_m v_2$ follows immediately from $\mathit{vvs}_1 \preceq_{\mathsf{s}}^{\delta^n} \mathit{vvs}_2$.