Proofs Regarding Specification of Spark Aggregation

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Laws regarding monads Recall the monad laws

$$f \mathbin{=\!\!\ll} \mathit{return}\; x = f\; x \quad , \tag{1}$$

$$\mathit{return} \mathbin{=\!\!\ll} m = m \quad , \tag{2}$$

$$f \mathbin{=\!\!\ll} (g \mathbin{=\!\!\ll} m) = (\lambda x \to f \mathbin{=\!\!\ll} g\; x) \mathbin{=\!\!\ll} m \quad . \tag{3}$$

Monadic application and composition are defined as:

$$(\mathbin{<\!\!=\!\!<}) :: (b \to m\; c) \to (a \to m\; b) \to a \to m\; c$$
$$(f \mathbin{<\!\!=\!\!<} g)\; x = f \mathbin{=\!\!\ll} g\; x$$
$$(\langle\$\rangle) :: (a \to b) \to m\; a \to m\; b$$
$$f \mathbin{\langle\$\rangle} m = (\mathit{return} \cdot f) \mathbin{=\!\!\ll} m$$
$$(\langle\bullet\rangle) :: (b \to c) \to (a \to m\; b) \to (a \to m\; c)$$
$$f \mathbin{\langle\bullet\rangle} g = ((\mathit{return} \cdot f) \mathbin{=\!\!\ll}) \cdot g$$

Laws concerning them include:

$$(f \mathbin{\langle\bullet\rangle} g)\; x = f \mathbin{\langle\$\rangle} g\; x \quad , \tag{4}$$

$$f \mathbin{=\!\!\ll} (g \mathbin{\langle\$\rangle} m) = (f \cdot g) \mathbin{=\!\!\ll} m \quad , \tag{5}$$

$$f \mathbin{\langle\$\rangle} (g \mathbin{\langle\$\rangle} m) = (f \cdot g) \mathbin{\langle\$\rangle} m \quad , \tag{6}$$

$$f \mathbin{\langle\bullet\rangle} (g \mathbin{\langle\bullet\rangle} m) = (f \cdot g) \mathbin{\langle\bullet\rangle} m \quad , \tag{7}$$

$$f \mathbin{<\!\!=\!\!<} (g \mathbin{\langle\bullet\rangle} h) = (f \cdot g) \mathbin{<\!\!=\!\!<} h \quad , \tag{8}$$

$$f \mathbin{\langle\bullet\rangle} (g \mathbin{<\!\!=\!\!<} h) = (f \mathbin{\langle\bullet\rangle} g) \mathbin{<\!\!=\!\!<} h \quad . \tag{9}$$

Proofs of the properties above will be given later, so as not to distract us from the main theorems.

Regarding monad-plus, we want $(\talloblong)$ to be associative, with *mzero* as identity. Monadic bind distributes into $(\talloblong)$ from the end:

$$f \mathbin{=\!\!\ll} (m \talloblong n) = (f \mathbin{=\!\!\ll} m) \talloblong (f \mathbin{=\!\!\ll} n) \quad . \tag{10}$$

It is less mentioned, but not uncommon, to demand that $(\talloblong)$ is also commutative and idempotent.

1 Folding and Shuffling

Lemma 1. *Given* $(\odot) :: a \to b \to b$*, we have*

$$\mathit{foldr}\ (\odot)\ z \mathbin{\langle\bullet\rangle} \mathit{insert}\ x = \mathit{return} \cdot \mathit{foldr}\ (\odot)\ z \cdot (x{:}) \quad ,$$

provided that $x \odot (y \odot z) = y \odot (x \odot z)$ *for all* $x, y :: a$ *and* $z :: b$.

Proof. Prove $\mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{insert}\ x\ xs = \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : xs))$. Induction on xs.
Case $xs := [\,]$.

$$\begin{array}{ll}
& \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{insert}\ x\ [\,] \\
= & \quad \{ \text{ definition of } (\langle\$\rangle) \ \} \\
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z) \mathbin{=\!\!\ll} \mathit{insert}\ x\ [\,] \\
= & \quad \{ \text{ definition of } \mathit{insert} \ \} \\
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z) \mathbin{=\!\!\ll} \mathit{return}\ [x] \\
= & \quad \{ \text{ monadic law (1) } \} \\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ [x]) \quad .
\end{array}$$

Case $xs := y : xs$.

$$\begin{array}{ll}
& \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{insert}\ x\ (y : xs) \\
= & \qquad \{ \text{ definition of } (\langle\$\rangle) \ \} \\
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z) \mathbin{=\!\!\ll} \mathit{insert}\ x\ (y : xs) \\
= & \qquad \{ \text{ definition of } \mathit{insert} \ \} \\
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z) \mathbin{=\!\!\ll} \\
& \quad (\mathit{return}\ (x : y : xs) \talloblong ((y{:}) \mathbin{\langle\$\rangle} \mathit{insert}\ x\ xs)) \\
= & \qquad \{ \text{ by (10) } \} \\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : y : xs)) \talloblong \\
& ((\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z \cdot (y{:})) \mathbin{=\!\!\ll} \mathit{insert}\ x\ xs) \quad .
\end{array}$$

Focus on the second branch:

$$\begin{array}{ll}
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z \cdot (y{:})) \mathbin{=\!\!\ll} \mathit{insert}\ x\ xs \\
= & \quad \{ \text{ definition of } \mathit{foldr} \ \} \\
& (\mathit{return} \cdot (y \odot) \cdot \mathit{foldr}\ (\odot)\ z) \mathbin{=\!\!\ll} \mathit{insert}\ x\ xs \\
= & \quad \{ \text{ by (5) } \} \\
& (\mathit{return} \cdot (y \odot)) \mathbin{=\!\!\ll} (\mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{insert}\ x\ xs) \\
= & \quad \{ \text{ induction } \}
\end{array}$$

$$
\begin{aligned}
& (\mathit{return} \cdot (y\odot)) =\!\!<\!\!< \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : \mathit{xs}))\\
= \quad & \{\ \text{monadic law (1)}\ \}\\
& \mathit{return}\ (y \odot \mathit{foldr}\ (\odot)\ z\ (x : \mathit{xs}))\\
= \quad & \{\ \text{definition of } \mathit{foldr}\ \}\\
& \mathit{return}\ (y \odot (x \odot \mathit{foldr}\ (\odot)\ z\ \mathit{xs}))\\
= \quad & \{\ \text{since } x \odot (y \odot z) = y \odot (x \odot z)\ \}\\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : y : \mathit{xs})) \quad .
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& (\mathit{foldr}\ (\odot)\ z \mathbin{\langle\bullet\rangle} \mathit{insert}\ x)\ (y : \mathit{xs})\\
= \quad & \{\ \text{calculatiion above}\ \}\\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : y : \mathit{xs})) \mathbin{[\!]} \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : y : \mathit{xs}))\\
= \quad & \{\ \text{idempotence of } ([\!])\ \}\\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : y : \mathit{xs})) \quad .
\end{aligned}
$$

□

Lemma 2. *Given* $(\odot) :: a \to b \to b$, *we have*

$$\mathit{foldr}\ (\odot)\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! = \mathit{return} \cdot \mathit{foldr}\ (\odot)\ z \quad ,$$

provided that $x \odot (y \odot z) = y \odot (x \odot z)$ *for all* $x, y :: a$ *and* $z :: b$.

Proof. Prove that $\mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{shuffle}\ \mathit{xs} = \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ \mathit{xs})$. Induction on xs.
Case $\mathit{xs} := [\,]$

$$
\begin{aligned}
& \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{shuffle}!\ [\,]\\
= \quad & \{\ \text{definitions of } (\langle\$\rangle) \text{ and } \mathit{shuffle}!\ \}\\
& (\mathit{return} \cdot \mathit{foldr}\ (\odot)\ z) =\!\!<\!\!< \mathit{return}\ [\,]\\
= \quad & \{\ \text{monadic law (1)}\ \}\\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ [\,]) \quad .
\end{aligned}
$$

Case $\mathit{xs} := x : \mathit{xs}$.

$$
\begin{aligned}
& \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{shuffle}!\ (x : \mathit{xs})\\
= \quad & \{\ \text{definition of } \mathit{shuffle}!\ \}\\
& \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} (\mathit{insert}\ x =\!\!<\!\!< \mathit{shuffle}\ \mathit{xs})\\
= \quad & \{\ \text{monadic law (3)}\ \}\\
& (\lambda \mathit{xs} \to \mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{insert}\ x\ \mathit{xs}) =\!\!<\!\!< \mathit{shuffle}!\ \mathit{xs}\\
= \quad & \{\ \text{Lemma 1}\ \}\\
& (\lambda \mathit{xs} \to \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : \mathit{xs}))) =\!\!<\!\!< \mathit{shuffle}!\ \mathit{xs}\\
= \quad & \{\ \text{definition of } \mathit{foldr}\ \}\\
& (\mathit{return} \cdot (x\odot) \cdot \mathit{foldr}\ (\odot)\ z) =\!\!<\!\!< \mathit{shuffle}\ \mathit{xs}\\
= \quad & \{\ \text{by (5)}\ \}\\
& (\mathit{return} \cdot (x\odot)) =\!\!<\!\!< (\mathit{foldr}\ (\odot)\ z \mathbin{\langle\$\rangle} \mathit{shuffle}\ \mathit{xs})\\
= \quad & \{\ \text{induction}\ \}\\
& (\mathit{return} \cdot (x\odot)) =\!\!<\!\!< (\mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ \mathit{xs}))
\end{aligned}
$$

$$
\begin{array}{ll}
= & \{\text{ monadic law (1) }\} \\
& \mathit{return}\ (x \odot \mathit{foldr}\ (\odot)\ z\ \mathit{xs}) \\
= & \{\text{ definition of } \mathit{foldr} \ \} \\
& \mathit{return}\ (\mathit{foldr}\ (\odot)\ z\ (x : \mathit{xs}))\quad .
\end{array}
$$

□

2 Map, Filter, and Shuffling

Lemma 3. $\mathit{insert}\ (f\ x) \cdot \mathit{map}\, f = \mathit{map}\, f \,\langle\bullet\rangle\, \mathit{insert}\ x$.

Proof. Prove that $\mathit{map}\, f \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs} = \mathit{insert}\ (f\ x)\ (\mathit{map}\, f\ \mathit{xs})$ for all xs, by induction on xs.
Case $\mathit{xs} := y : \mathit{xs}$.

$$
\begin{array}{ll}
& \mathit{map}\, f \,\langle\$\rangle\, \mathit{insert}\ x\ (y : \mathit{xs}) \\
= & \{\text{ definition of } \mathit{insert} \ \} \\
& \mathit{map}\, f \,\langle\$\rangle\, (\mathit{return}\ (x : y : \mathit{xs}) \mathbin{[\!]} ((y{:}) \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs})) \\
= & \{\text{ (10), definition of } (\langle\$\rangle) \ \} \\
& (\mathit{map}\, f \,\langle\$\rangle\, \mathit{return}\ (x : y : \mathit{xs})) \mathbin{[\!]} (\mathit{map}\, f \,\langle\$\rangle\, ((y{:}) \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs}))\quad .
\end{array}
$$

The first branch, by definition of $(\langle\$\rangle)$ and monadic law (1), simplifies to $\mathit{return}\ (\mathit{map}\, f\ (x : y : \mathit{xs}))$. The second branch:

$$
\begin{array}{ll}
& \mathit{map}\, f \,\langle\$\rangle\, ((y{:}) \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs}) \\
= & \{\text{ (6) }\} \\
& (\mathit{map}\, f \cdot (y{:})) \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs} \\
= & \{\text{ definition of } \mathit{map} \ \} \\
& ((f\ y{:}) \cdot \mathit{map}\, f) \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs} \\
= & \{\text{ (6) }\} \\
& (f\ y{:}) \,\langle\$\rangle\, (\mathit{map}\, f \,\langle\$\rangle\, \mathit{insert}\ x\ \mathit{xs}) \\
= & \{\text{ induction }\} \\
& (f\ y{:}) \,\langle\$\rangle\, (\mathit{insert}\ (f\ x)\ (\mathit{map}\, f\ \mathit{xs}))\quad .
\end{array}
$$

Thus we have

$$
\begin{array}{ll}
& \mathit{map}\, f \,\langle\$\rangle\, \mathit{insert}\ x\ (y : \mathit{xs}) \\
= & \{\text{ calculation above }\} \\
& \mathit{return}\ (f\ x : f\ y : \mathit{map}\, f\ \mathit{xs}) \mathbin{[\!]} ((f\ y{:}) \,\langle\$\rangle\, (\mathit{insert}\ (f\ x)\ (\mathit{map}\, f\ \mathit{xs}))) \\
= & \{\text{ definition of } \mathit{insert} \ \} \\
& \mathit{insert}\ (f\ x)\ (f\ y : \mathit{map}\, f\ \mathit{xs}) \\
= & \{\text{ definition of } \mathit{map} \ \} \\
& \mathit{insert}\ (f\ x)\ (\mathit{map}\, f\ (y : \mathit{xs}))\quad .
\end{array}
$$

□

Lemma 4. $\mathit{shuffle}! \cdot \mathit{map}\, f = \mathit{map}\, f \,\langle\bullet\rangle\, \mathit{shuffle}!$.

Lemma 5. $\mathit{shuffle}! \cdot \mathit{filter}\ p = \mathit{filter}\ p \,\langle\bullet\rangle\, \mathit{shuffle}!$.

3 Homomorphism, etc

Lemma 6. *$h = hom\ (\oplus)\ z$ if and only if $foldr\ (\oplus)\ z \cdot map\ h = h \cdot concat$.*

Proof. A Ping-pong proof.
Direction ($\Rightarrow$). Prove $foldr\ (\oplus)\ z\ (map\ h\ xss) = h\ (concat\ xss)$ by induction on xss.
Case $xss := [\,]$:

$$
\begin{aligned}
& foldr\ (\oplus)\ z\ (map\ h\ [\,]) \\
= {} & foldr\ (\oplus)\ z\ [\,] \\
= {} & z \\
= {} & h\ (concat\ [\,]) \quad .
\end{aligned}
$$

Case $xss := xs : xss$:

$$
\begin{aligned}
& foldr\ (\oplus)\ z\ (map\ h\ (xs : xss)) \\
= {} & h\ xs \oplus foldr\ (\oplus)\ z\ (map\ h\ xss) \\
= {} & \quad \{ \text{ induction } \} \\
& h\ xs \oplus h\ (concat\ xss) \\
= {} & \quad \{ \ h \text{ homomorphism } \} \\
& h\ (concat\ (xs : xss)) \quad .
\end{aligned}
$$

Direction ($\Leftarrow$). Show that h satisfies the properties being a list homomorphism. On empty list:

$$
\begin{aligned}
& h\ [\,] \\
= {} & h\ (concat\ [\,]) \\
= {} & \quad \{ \text{ assumption } \} \\
& foldr\ (\oplus)\ z\ (map\ h\ [\,]) \\
= {} & z \quad .
\end{aligned}
$$

On concatentation:

$$
\begin{aligned}
& h\ (xs +\!\!+ ys) \\
= {} & h\ (concat\ [xs, ys]) \\
= {} & \quad \{ \text{ assumption } \} \\
& foldr\ (\oplus)\ z\ (map\ h\ [xs, ys]) \\
= {} & h\ xs \oplus (h\ ys \oplus z) \\
= {} & h\ xs \oplus h\ ys \quad .
\end{aligned}
$$

□

Lemma 7. *Let $(\oplus) :: b \to b \to b$ be associative on $img\ (foldr\ (\otimes)\ z)$ with z as its identity, where $(\otimes) :: a \to b \to b$. We have $foldr\ (\otimes)\ z = hom\ (\oplus)\ z$ if and only if $x \otimes (y \oplus w) = (x \otimes y) \oplus w$ for all $x :: a$ and $y, w \in img\ (foldr\ (\otimes)\ z)$.*

Proof. A Ping-pong proof.

Direction ($\Leftarrow$). We show that $\mathit{foldr}\ (\otimes)\ z$ satisfies the homomorphic properties. It is immediate that $\mathit{foldr}\ (\otimes)\ z\ [\,] = z$. For $xs +\!\!+\, ys$, note that

$$\mathit{foldr}\ (\otimes)\ z\ (xs +\!\!+\, ys) = \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ ys)\ xs \quad .$$

The aim is thus to prove that

$$\mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ ys)\ xs = (\mathit{foldr}\ (\otimes)\ z\ xs) \oplus (\mathit{foldr}\ (\otimes)\ z\ ys) \quad .$$

We perform an induction on xs. The case when $xs := [\,]$ trivially holds. For $xs := x : xs$, we reason:

$$\begin{aligned}
& \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ ys)\ (x : xs) \\
=\ & x \otimes \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ ys)\ xs \\
=\ & \quad \{\ \text{induction}\ \} \\
& x \otimes ((\mathit{foldr}\ (\otimes)\ z\ xs) \oplus (\mathit{foldr}\ (\otimes)\ z\ ys)) \\
=\ & \quad \{\ \text{assumption: } x \otimes (y \oplus w) = (x \otimes y) \oplus w\ \} \\
& (x \otimes (\mathit{foldr}\ (\otimes)\ z\ xs)) \oplus (\mathit{foldr}\ (\otimes)\ z\ ys) \\
=\ & (\mathit{foldr}\ (\otimes)\ z\ (x : xs)) \oplus (\mathit{foldr}\ (\otimes)\ z\ ys) \quad .
\end{aligned}$$

Direction ($\Rightarrow$). Given $\mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z$. Let $y = \mathit{foldr}\ (\otimes)\ z\ xs$ and $w = \mathit{foldr}\ (\otimes)\ z\ ys$ for some xs and ys. We reason:

$$\begin{aligned}
& x \otimes (y \oplus w) \\
=\ & x \otimes (\mathit{foldr}\ (\otimes)\ z\ xs \oplus \mathit{foldr}\ (\otimes)\ z\ ys) \\
=\ & \quad \{\ \text{since } \mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z\ \} \\
& x \otimes (\mathit{foldr}\ (\otimes)\ z\ (xs +\!\!+\, ys)) \\
=\ & \mathit{foldr}\ (\otimes)\ z\ (x : xs +\!\!+\, ys) \\
& \quad \{\ \text{since } \mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z\ \} \\
=\ & \mathit{foldr}\ (\otimes)\ z\ (x : xs) \oplus \mathit{foldr}\ (\otimes)\ z\ ys \\
=\ & (x \otimes \mathit{foldr}\ (\otimes)\ z\ xs) \oplus \mathit{foldr}\ (\otimes)\ z\ ys \\
=\ & (x \otimes y) \oplus w \quad .
\end{aligned}$$

□

4 Aggregation

Lemma 8. *Given* $(\otimes) :: a \to b \to b$ *and define* $xs \odot w = \mathit{foldr}\ (\otimes)\ w\ xs$. *We have* $\mathit{foldr}\ (\otimes)\ z \cdot \mathit{concat} = \mathit{foldr}\ (\odot)\ z$.

Proof. By *foldr* fusion the proof obligation is

$$\mathit{foldr}\ (\otimes)\ z\ (xs +\!\!+\, ys) = \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ z\ ys)\ xs \quad .$$

Induction on xs.
Case $xs := [\,]$:

$$
\begin{aligned}
& \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ z\ \mathit{ys})\ [\,] \\
={}& \mathit{foldr}\ (\otimes)\ z\ \mathit{ys} \\
={}& \mathit{foldr}\ (\otimes)\ z\ ([\,] +\!\!+\ \mathit{ys}) \quad .
\end{aligned}
$$

Case $\mathit{xs} := x : \mathit{xs}$:

$$
\begin{aligned}
& \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ z\ \mathit{ys})\ (x : \mathit{xs}) \\
={}& x \otimes \mathit{foldr}\ (\otimes)\ (\mathit{foldr}\ (\otimes)\ z\ \mathit{ys})\ \mathit{xs} \\
={}& \quad \{ \text{ induction } \} \\
& x \otimes \mathit{foldr}\ (\otimes)\ z\ (\mathit{xs} +\!\!+\ \mathit{ys}) \\
={}& \mathit{foldr}\ (\otimes)\ z\ (x : \mathit{xs} +\!\!+\ \mathit{ys}) \quad .
\end{aligned}
$$

□

Theorem 9. *$\mathit{aggregate}\ z\ (\otimes)\ (\oplus) = \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)$, provided that $(\oplus)$ is associative, commutative, and has z as identity.*

Proof. We reason:

$$
\begin{aligned}
& \mathit{aggregate}\ z\ (\otimes)\ (\oplus) \\
={}& \quad \{ \text{ definition of } \mathit{aggregate} \ \} \\
& \mathit{foldr}\ (\oplus)\ z \mathbin{\langle\bullet\rangle} (\mathit{shuffle}! \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)) \\
={}& \quad \{ \text{ Lemma 4 } \} \\
& \mathit{foldr}\ (\oplus)\ z \mathbin{\langle\bullet\rangle} (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \mathbin{\langle\bullet\rangle} \mathit{shuffle}!) \\
={}& \quad \{ \text{ by (7) } \} \\
& (\mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)) \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \\
={}& \quad \{ \text{ Lemma 2 } \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \quad .
\end{aligned}
$$

The last step holds because by *foldr*-*map* fusion, for all h,

$$
\begin{aligned}
& \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ h = \mathit{foldr}\ (\odot)\ z \\
& \quad \textbf{where}\ \mathit{xs} \odot w = h\ \mathit{xs} \oplus w \quad ,
\end{aligned}
$$

and $(\odot)$ satisfies that $\mathit{xs} \odot (\mathit{ys} \odot w) = \mathit{ys} \odot (\mathit{xs} \odot w)$ if $(\oplus)$ is associative, commutative, and has z as identity. □

Corollary 10. *$\mathit{aggregate}\ z\ (\otimes)\ (\oplus) = \mathit{return} \cdot \mathit{foldr}\ (\otimes)\ z \cdot \mathit{concat}$, provided that $(\oplus)$ is associative, commutative, and has z as identity, and that $\mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z$.*

Proof. We reason:

$$
\begin{aligned}
& \mathit{aggregate}\ z\ (\otimes)\ (\oplus) \\
={}& \quad \{ \text{ Theorem 9 } \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \\
={}& \quad \{ \text{ Lemma 6 } \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\otimes)\ z \cdot \mathit{concat} \quad .
\end{aligned}
$$

□

Lemma 11. *If $aggregate\ z\ (\otimes)\ (\oplus) = return \cdot foldr\ (\otimes)\ z \cdot concat$, and $shuffle!\ xss = return\ yss \mathbin{[\!]} m$, we have*

$$\begin{array}{l} foldr\ (\otimes)\ z\ (concat\ xss) = \\ \quad foldr\ (\oplus)\ z\ (map\ (foldr\ (\otimes)\ z)\ xss) = \\ \qquad foldr\ (\oplus)\ z\ (map\ (foldr\ (\otimes)\ z)\ yss) \quad . \end{array}$$

Proof. We assume the following two properties of MonadPlus:

1. $m_1 \mathbin{[\!]} m_2 = return\ x$ implies that $m_1 = m_2 = return\ x$.
2. $return\ x_1 = return\ x_2$ implies that $x_1 = x_2$.

For our problem, if $shuffle!\ xss = return\ yss \mathbin{[\!]} m$, we have

$$\begin{array}{ll} & return \cdot foldr\ (\otimes)\ z \cdot concat\ \$\ xss \\ = & \quad \{\ \text{assumption}\ \} \\ & aggregate\ z\ (\otimes)\ (\oplus)\ \$\ xss \\ = & \quad \{\ \text{calculation in the previous lemma}\ \} \\ & (foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z))\ \langle\$\rangle\ shuffle!\ xss \\ = & \quad \{\ shuffle!\ xss = return\ yss \mathbin{[\!]} m\ \} \\ & (return \cdot foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z)\ \$\ yss) \mathbin{[\!]} \\ & \quad ((foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z))\ \langle\$\rangle\ m) \quad . \end{array}$$

□

Theorem 12. *If $aggregate\ z\ (\otimes)\ (\oplus) = return \cdot foldr\ (\otimes)\ z \cdot concat$, we have that $(\oplus)$, when restricted to values in $img\ (foldr\ (\otimes)\ z)$, is associative, commutative, and has z as identity.*

Proof. In the discussion below, let x,y, and w be in $img\ (foldr\ (\otimes)\ z)$. That is, there exists xs, ys, and ws such that $x = foldr\ (\otimes)\ z\ xs$, $y = foldr\ (\otimes)\ z\ ys$, and $w = foldr\ (\otimes)\ z\ ws$.
Identity. We reason:

$$\begin{array}{ll} & y \\ = & foldr\ (\otimes)\ z\ (concat\ [xs]) \\ = & \quad \{\ shuffle!\ [xs] = return\ [xs] \mathbin{[\!]} mzero, \text{Lemma 11}\ \} \\ & foldr\ (\oplus)\ z\ (map\ (foldr\ (\otimes)\ z)\ [xs]) \\ = & y \oplus z \quad . \end{array}$$

Thus z is a right identity of $(\oplus)$.

$$\begin{array}{ll} & y \\ = & foldr\ (\otimes)\ z\ (concat\ [[\,], xs]) \\ = & \quad \{\ shuffle!\ [[\,], xs] = return\ [[\,], xs] \mathbin{[\!]} m, \text{Lemma 11}\ \} \\ & foldr\ (\oplus)\ z\ (map\ (foldr\ (\otimes)\ z)\ [[\,], xs]) \\ = & z \oplus (y \oplus z) \\ = & \quad \{\ z \text{ is a right identity of } (\oplus)\ \} \\ & z \oplus y \quad . \end{array}$$

Thus z is also a left identity of $(\oplus)$.
Commutativity. We reason:

$$
\begin{aligned}
& x \oplus y \\
= \quad & \{ z \text{ is a right identity } \} \\
& x \oplus (y \oplus z) \\
= \; & \mathit{foldr}\ (\oplus)\ z\ (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)\ [xs, ys]) \\
= \quad & \{ \mathit{shuffle}\ [xs, ys] = \mathit{return}\ [ys, xs] \mathbin{[\!]} m, \text{ Lemma 11 } \} \\
& \mathit{foldr}\ (\oplus)\ z\ (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)\ [ys, xs]) \\
= \; & y \oplus (x \oplus z) \\
& y \oplus x \quad .
\end{aligned}
$$

Associativity. We reason:

$$
\begin{aligned}
& x \oplus (y \oplus w) \\
= \quad & \{ z \text{ is a right identity } \} \\
& x \oplus (y \oplus (w \oplus z)) \\
= \; & \mathit{foldr}\ (\oplus)\ z\ (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)\ [xs, ys, ws]) \\
= \quad & \{ \ \} \\
& \mathit{foldr}\ (\oplus)\ z\ (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)\ [ws, xs, ys]) \\
= \; & w \oplus (x \oplus (y \oplus z)) \\
= \quad & \{ z \text{ is a right identity } \} \\
& w \oplus (x \oplus y) \\
= \quad & \{ (\oplus) \text{ commutative } \} \\
& (x \oplus y) \oplus w \quad .
\end{aligned}
$$

□

Theorem 13. *If* $\mathit{aggregate}\ z\ (\otimes)\ (\oplus) = \mathit{return} \cdot \mathit{foldr}\ (\otimes)\ z \cdot \mathit{concat}$*, we have* $\mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z$*.*

Proof. Apparently $\mathit{foldr}\ (\otimes)\ z\ [\,] = z$. We are left with proving the case for concatenation.

$$
\begin{aligned}
& \mathit{foldr}\ (\otimes)\ z\ (xs +\!\!+ ys) \\
= \; & \mathit{foldr}\ (\otimes)\ z\ (\mathit{concat}\ [xs, ys]) \\
= \quad & \{ \text{ Lemma 11 } \} \\
& \mathit{foldr}\ (\oplus)\ z\ (\mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)\ [xs, ys]) \\
= \; & \mathit{foldr}\ (\otimes)\ z\ xs \oplus (\mathit{foldr}\ (\oplus)\ z\ ys \oplus z) \\
= \quad & \{ \text{ Theorem 12, } z \text{ is identity } \} \\
& \mathit{foldr}\ (\otimes)\ z\ xs \oplus \mathit{foldr}\ (\oplus)\ z\ ys \quad .
\end{aligned}
$$

□

Corollary 14. *Given* $(\otimes) :: a \to b \to b$ *and* $(\oplus) :: b \to b \to b$*.* $\mathit{aggregate}\ z\ (\otimes)\ (\oplus) = \mathit{return} \cdot \mathit{foldr}\ (\otimes)\ z \cdot \mathit{concat}$ *if and only if* $(\mathit{img}\ (\mathit{foldr}\ (\otimes)\ z), (\oplus), z)$ *forms a commutative monoid, and that* $\mathit{foldr}\ (\otimes)\ z = \mathit{hom}\ (\oplus)\ z$*.*

Proof. A conclusion following from Corollary 10, Theorem 12, and Theorem 13. □

5 Tree Aggregate

Lemma 15. *$apply!\ (\oplus) = return \cdot foldr\ (\oplus)\ z$ if $(\oplus)$ is associative, commutative, and has z as identity.*

Proof. To do. □

Theorem 16. *$treeAggregate\ z\ (\otimes)\ (\oplus) = return \cdot foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z)$ if $(\oplus)$ is associative, commutative, and has z as identity.*

Proof.

$$
\begin{array}{ll}
& treeAggregate\ z\ (\otimes)\ (\oplus) \\
= & \{ \text{definition of } treeAggregate \} \\
& apply!\ (\oplus) \lll (shuffle! \cdot map\ (foldr\ (\otimes)\ z)) \\
= & \{ \text{Lemma 4} \} \\
& apply!\ (\oplus) \lll (map\ (foldr\ (\otimes)\ z) \odot shuffle!) \\
= & \{ \text{by (8)} \} \\
& (apply!\ (\oplus) \cdot map\ (foldr\ (\otimes)\ z)) \lll shuffle! \\
= & \{ \text{Lemma 15} \} \\
& (return \cdot foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z)) \lll shuffle! \\
= & \{ \text{definitions of } (\lll) \text{ and } (\odot) \} \\
& (foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z)) \odot shuffle! \\
= & \{ \text{Lemma 2} \} \\
& return \cdot foldr\ (\oplus)\ z \cdot map\ (foldr\ (\otimes)\ z) \quad .
\end{array}
$$

□

6 Aggregation by Key

For proofs, it is helpful to have an inductive definition of $(\hat{\odot}_z)$.

$$
\begin{array}{l}
(k,v)\ \hat{\odot}_z\ [\,] = [(k, v \odot z)] \\
(k,v)\ \hat{\odot}_z\ ((j,u) : xs) \\
\quad \mid k == j \qquad\ \ = (k, v \odot u) : xs \\
\quad \mid otherwise = (j,u) : ((k,v)\ \hat{\odot}_z\ xs) \quad .
\end{array}
$$

Define the following auxiliary functions:

$$
\begin{array}{l}
kEq\ k = (k ==) \cdot fst \quad , \\
lookUp \quad :: \mathsf{Eq}\ k \Rightarrow k \to a \to \mathsf{PairRDD}\ k\ a \to a \\
lookUp\ k\ z = hd\ z \cdot map\ snd \cdot filter\ (keyEq\ k) \cdot concat \quad , \\
filterKRDD \quad :: \mathsf{Eq}\ k \Rightarrow k \to \mathsf{PairRDD}\ k\ a \to \mathsf{PairRDD}\ k\ a \\
filterKRDD\ k = filter\ (\neg \cdot null) \cdot map\ (filter\ (kEq\ k)) \quad .
\end{array}
$$

Lemma 17. *For all j, k, xs, v, z, and $(\odot)$, we have:*

1. *$filter\ (kEq\ k)\ ((k,v)\ \hat{\odot}_z\ xs) = (k,v)\ \hat{\odot}_z\ filter\ (kEq\ k)\ xs$.*

2. $\mathit{filter}\ (\mathit{kEq}\ k)\ ((j,v)\ \hat{\odot}_z\ \mathit{xs}) = \mathit{filter}\ (\mathit{kEq}\ k)\ \mathit{xs}$, *if* $k \not\equiv j$.

Lemma 18. *For all* k, $(\oplus)$, *and* z, *we have*

$$\mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] = \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{filter}\ (\mathit{kEq}\ k) \quad .$$

Lemma 19. *For all* k, $(\odot)$, *and* z, *we have*

$$\begin{array}{l} \mathit{foldr}\ (\hat{\odot}_z)\ [\,]\ (\mathit{filter}\ (\mathit{kEq}\ k)\ \mathit{xs}) = \\ \quad [(k, \mathit{foldr}\ (\odot)\ z \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k)\ \$\ \mathit{xs})] \end{array}$$

if there exists at least one (k,v) *in* xs. *Otherwise* $\mathit{foldr}\ (\hat{\odot}_z)\ [\,]\ (\mathit{filter}\ (\mathit{kEq}\ k)\ \mathit{xs}) = [\,]$.

Corollary 20. *As a corollary, we have that for all* k, $(\odot)$, *and* z,

$$\begin{array}{l} \mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \cdot \mathit{map}\ \mathit{value} \cdot \mathit{foldr}\ (\hat{\odot}_z)\ [\,] \cdot \mathit{filter}\ (\mathit{kEq}\ k) = \\ \quad \mathit{return} \cdot \mathit{foldr}\ \mathit{dot}\ z \cdot \mathit{map}\ \mathit{value} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \quad . \end{array}$$

It follows from Lemma 19 because, when the input is empty or does not contain entries with key k, both sides reduce to $\mathit{return}\ z$.

Corollary 21. *For all* k, z *and* $(\odot)$ *we have:*

$$\begin{array}{l} \mathit{concat} \cdot \mathit{map}\ (\mathit{map}\ \mathit{value} \cdot \mathit{filter}\ (\mathit{keyEq}\ k) \cdot \mathit{foldr}\ (\hat{\odot}_z)\ [\,]) = \\ \quad \mathit{map}\ (\mathit{foldr}\ (\odot)\ z) \cdot \mathit{filterKRDD}\ k \quad . \end{array}$$

Theorem 22. *If ..., we have*

$$\begin{array}{l} \mathit{lookUp}\ k\ z \mathbin{\langle\bullet\rangle} \mathit{aggregateByKey}\ z\ (\otimes)\ (\oplus) = \\ \quad \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \cdot \mathit{filterKRDD}\ k \quad . \end{array}$$

Proof. We reason:

$$\begin{array}{ll}
 & \mathit{lookUp}\ k\ z \mathbin{\langle\bullet\rangle} \mathit{aggregateByKey}\ z\ (\otimes)\ (\oplus) \\
= & \quad \{\ \text{definitions}\ \} \\
 & (\mathit{hd}\ z \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{concat}) \mathbin{\langle\bullet\rangle} \\
 & \mathit{repartition}! \lll (\mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat}) \mathbin{\langle\bullet\rangle} \mathit{map}!\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{\ \mathit{concat} \mathbin{\langle\bullet\rangle} \mathit{part} = \mathit{return}\ \} \\
 & (\mathit{hd}\ z \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k)) \mathbin{\langle\bullet\rangle} \\
 & \mathit{shuffle}! \lll (\mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat}) \mathbin{\langle\bullet\rangle} \mathit{map}!\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{\ \text{Lemma 4 and 5}\ \} \\
 & \mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \lll (\mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat}) \mathbin{\langle\bullet\rangle} \mathit{map}!\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{\ \text{Lemma 4}\ \} \\
 & \mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \lll (\mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,])) \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \\
= & \quad \{\ \text{by (8) and (9)}\ \} \\
 & (\mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,])) \lll \mathit{shuffle}! \quad .
\end{array}$$

We work on the part before the right-most *shuffle*!:

$$
\begin{aligned}
& \mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{ \text{Lemma 18} \} \\
& \mathit{hd}\ z \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{foldr}\ (\hat{\oplus}_z)\ [\,] \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{ \text{Corollary 20} \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{ \text{naturalty} \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{concat} \cdot \mathit{map}\ (\mathit{map}\ \mathit{snd} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{foldr}\ (\hat{\otimes}_z)\ [\,]) \\
= & \quad \{ \text{Corollary 21} \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \cdot \mathit{filterKRDD}\ k \quad .
\end{aligned}
$$

Back to the main proof:

$$
\begin{aligned}
& \mathit{lookUp}\ k\ z \mathbin{\langle\bullet\rangle} \mathit{aggregateByKey}\ z\ (\otimes)\ (\oplus) \\
= & \quad \{ \text{calculation above} \} \\
& (\mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \cdot \mathit{filter}\ (\neg \cdot \mathit{null}) \cdot \mathit{map}\ (\mathit{filter}\ (\mathit{kEq}\ k))) \lll \mathit{shuffle}! \\
= & \quad \{ \text{definitions of } (\lll) \text{ and } (\langle\bullet\rangle) \} \\
& (\mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \cdot \mathit{filter}\ (\neg \cdot \mathit{null}) \cdot \mathit{map}\ (\mathit{filter}\ (\mathit{kEq}\ k))) \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \\
= & \quad \{ \text{Lemma 4 and 5} \} \\
& (\mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z)) \mathbin{\langle\bullet\rangle} \mathit{shuffle}! \cdot \mathit{filterKRDD}\ k \\
= & \quad \{ \text{Lemma 2} \} \\
& \mathit{return} \cdot \mathit{foldr}\ (\oplus)\ z \cdot \mathit{map}\ (\mathit{foldr}\ (\otimes)\ z) \cdot \mathit{filterKRDD}\ k \quad .
\end{aligned}
$$

□

$$
\begin{aligned}
& \mathit{lookUp}\ k\ z \mathbin{\langle\$\rangle} (\mathit{aggregateMessagesWithActiveVertices} \\
& \quad \mathit{sendMsg}\ (\oplus)\ \mathit{active}\ (\mathsf{Graph}\ \mathit{vRdd}\ \mathit{eRdd})) \\
= & \quad \{ \text{definitions} \} \\
& \mathit{lookUp}\ k\ z \mathbin{\langle\$\rangle} \mathit{reduceByKey}\ (\oplus)\ (\mathit{map}\ (\mathit{concatMap}\ \mathit{sendIfActive})\ \mathit{eRdd}) \\
= & \quad \{ \} \\
& \mathit{return} \cdot \mathit{foldl}\ (\oplus)\ z \cdot \mathit{concat} \cdot \mathit{filterKRDD}\ k \cdot \\
& \quad \mathit{map}\ (\mathit{concatMap}\ \mathit{sendIfActive})\ \$\ \mathit{eRdd} \\
= & \quad \{ \text{naturalty laws} \} \\
& \mathit{return} \cdot \mathit{foldl}\ (\oplus)\ z \cdot \mathit{concatMap}\ \mathit{sendIfActive} \cdot \\
& \quad \mathit{map}\ \mathit{value} \cdot \mathit{filter}\ (\mathit{kEq}\ k) \cdot \mathit{concat}\ \$\ \mathit{eRdd} \quad .
\end{aligned}
$$

7 Proofs of monadic properties

Proving (5) $\quad f \mathrel{=\!\!\lll} (g \mathbin{\langle\$\rangle} m) = (f \cdot g) \mathrel{=\!\!\lll} m.$

Proof. We reason:

$$
\begin{aligned}
& f \mathrel{=\!\!\lll} (g \mathbin{\langle\$\rangle} m) \\
= & \quad \{ \text{definition of } (\langle\$\rangle) \}
\end{aligned}
$$

$$
\begin{aligned}
& f =\!\!\ll ((\mathit{return} \cdot g) =\!\!\ll m) \\
= & \quad \{ \text{ monadic law (3) } \} \\
& (\lambda x \to f =\!\!\ll \mathit{return}\ (g\ x)) =\!\!\ll m \\
= & \quad \{ \text{ monadic law (1) } \} \\
& (\lambda x \to f\ (g\ x)) =\!\!\ll m \\
= & \ (f \cdot g) =\!\!\ll m \quad .
\end{aligned}
$$

□

Proving (6) $f \,\langle\$\rangle\, (g \,\langle\$\rangle\, m) = (f \cdot g) \,\langle\$\rangle\, m$.

Proof. We reason:

$$
\begin{aligned}
& f \,\langle\$\rangle\, (g \,\langle\$\rangle\, m) \\
= & \quad \{ \text{ definition of } (\langle\$\rangle) \ \} \\
& (\mathit{return} \cdot f) =\!\!\ll (g \,\langle\$\rangle\, m) \\
= & \quad \{ \text{ by (5) } \} \\
& (\mathit{return} \cdot f \cdot g) =\!\!\ll m \\
= & \quad \{ \text{ definition of } (\langle\$\rangle) \ \} \\
& (f \cdot g) =\!\!\ll m \quad .
\end{aligned}
$$

□

For the next results we prove a lemma:

$$
(f =\!\!\ll) \cdot (g =\!\!\ll) = (((f =\!\!\ll) \cdot g) =\!\!\ll) \quad . \tag{11}
$$

$$
\begin{aligned}
& (f =\!\!\ll) \cdot (g =\!\!\ll) \\
= & \quad \{ \ \eta \text{ intro. } \} \\
& (\lambda m \to f =\!\!\ll (g =\!\!\ll m)) \\
= & \quad \{ \text{ monadic law (3) } \} \\
& (\lambda m \to (\lambda y \to f =\!\!\ll g\ y) =\!\!\ll m) \\
= & \quad \{ \ \eta \text{ reduction } \} \\
& (((f =\!\!\ll) \cdot g) =\!\!\ll) \quad .
\end{aligned}
$$

Proving (7) $f \,\langle\bullet\rangle\, (g \,\langle\bullet\rangle\, m) = (f \cdot g) \,\langle\bullet\rangle\, m$.

Proof. We reason:

$$
\begin{aligned}
& f \,\langle\bullet\rangle\, (g \,\langle\bullet\rangle\, m) \\
= & \quad \{ \text{ definition of } (\langle\bullet\rangle) \ \} \\
& ((\mathit{return} \cdot f) =\!\!\ll) \cdot ((\mathit{return} \cdot g) =\!\!\ll) \cdot m \\
= & \quad \{ \text{ by (11) } \} \\
& ((((\mathit{return} \cdot f) =\!\!\ll) \cdot \mathit{return} \cdot g) =\!\!\ll) \cdot m \\
= & \quad \{ \text{ monadic law (1) } \} \\
& ((\mathit{return} \cdot f \cdot g) =\!\!\ll) \cdot m \\
= & \quad \{ \text{ definition of } (\langle\bullet\rangle) \ \} \\
& (f \cdot g) \,\langle\bullet\rangle\, m \quad .
\end{aligned}
$$

□

Proving (8) $f \mathrel{<\!\!=\!\!<} (g \mathbin{\langle\bullet\rangle} h) = (f \cdot g) \mathrel{<\!\!=\!\!<} h$.

Proof. We reason:

$$\begin{array}{ll}
 & f \mathrel{<\!\!=\!\!<} (g \mathbin{\langle\bullet\rangle} h) \\
= & \quad \{ \text{ definitions of } (\mathrel{<\!\!=\!\!<}) \text{ } \} \\
 & (f {=\!\!\ll}) \cdot ((\mathit{return} \cdot g) {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ by (11) } \} \\
 & (((f {=\!\!\ll}) \cdot \mathit{return} \cdot g) {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ monadic law (1) } \} \\
 & (((f \cdot g) {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ definition of } (\mathrel{<\!\!=\!\!<}) \text{ } \} \\
 & (f \cdot g) \mathrel{<\!\!=\!\!<} h \quad .
\end{array}$$

□

Proving (9) $f \mathbin{\langle\bullet\rangle} (g \mathrel{<\!\!=\!\!<} h) = (f \mathbin{\langle\bullet\rangle} g) \mathrel{<\!\!=\!\!<} h$.

Proof. We reason:

$$\begin{array}{ll}
 & f \mathbin{\langle\bullet\rangle} (g \mathrel{<\!\!=\!\!<} h) \\
= & \quad \{ \text{ definitions of } (\mathrel{<\!\!=\!\!<}) \text{ and } (\langle\bullet\rangle) \text{ } \} \\
 & ((\mathit{return} \cdot f) {=\!\!\ll}) \cdot (g {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ by (11) } \} \\
 & ((((\mathit{return} \cdot f) {=\!\!\ll}) \cdot g) {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ definition of } (\langle\bullet\rangle) \text{ } \} \\
 & ((f \mathbin{\langle\bullet\rangle} g) {=\!\!\ll}) \cdot h \\
= & \quad \{ \text{ definition of } (\mathrel{<\!\!=\!\!<}) \text{ } \} \\
 & (f \mathbin{\langle\bullet\rangle} g) \mathrel{<\!\!=\!\!<} h \quad .
\end{array}$$

□